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Oscillations of an inhomogeneous bounded plasma

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Quasi-static waves on a plasma cylinder with radially varying density have been studied. An approximate dispersion relation is derived, with the use of a di-electric tensor which is a local quantity, and is compared with the case of a homogeneous plasma cylinder.

Introduction

The study of oscillations of a plasma cylinder is of interest as this configuration has been used in several laboratory experiments (Apel 1969, Barrett *et al* 1968, Wong *et al* 1964) The case of homogeneous magnetoactive plasma confined within a cylindrical conductor has been studied by Lichtenberg & Jayson (1965). These authors have used a di-electric tensor which is correct within the quasi-static approximation and approximation of longitudinal velocity only. Inhomogeneous plasma in the same configuration has been considered by Lee (1969). Here the same problem of inhomogeneous plasma in a metallic cylinder is treated, but with the use of di-electric tensor whose local value is of the form used by Lichtenberg & Jayson, with this exception that only one component plasma is considered and the velocity of streaming neglected. The dispersion relation is derived using quasi-static approximation and reduced for the case of cold plasma also. The results have been compared with the case of a homogeneous bounded plasma.

Derivation of the dispersion relation

A plasma cylinder of radius b is surrounded by a co-axial cylindrical conducting wall of radius a. The cross-sectional variation of plasma density is assumed as

$$n(r) = n_0 f(r) \qquad \ldots \quad (1)$$

with $f(r) = 1 - r^2/\lambda_1^2$, so that n_0 is the density on the axis $r = 0$, of the cylinder. There is a steady magnetic field H along this axis.

The di-electric tensor used in this analysis is of the form

$$\bar{\epsilon} = \begin{bmatrix} \epsilon_{rr} & i\epsilon_{r\theta} & 0 \\ -i\epsilon_{r\theta} & \epsilon_{\theta\theta} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix} \qquad \ldots \quad (2)$$

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with
$$\left.\begin{aligned}
\epsilon_{rr} &= 1+\frac{\omega_p^2 f(r)}{2^{3/2}K_z\omega_H v_T}[z(\zeta_2)-z(\zeta_1)] = 1+\tau f(r)\\
\epsilon_{r\theta} &= \frac{\omega_p^2 f(r)}{2^{3/2}K_z\omega_H v_T}[2z(\zeta_3)-z(\zeta)_2-z(\zeta_1)]\\
\epsilon_{zz} &= 1+\frac{\omega_p^2 f(r)}{K_z^2 v_T^2}[1+\zeta_3 z(\zeta_3)] = 1+\tau_0 f(r)
\end{aligned}\right\} \quad \ldots \quad (2A)$$

where $z(\zeta) = 2ie^{-\zeta^2}\int_{-\infty}^{i\zeta} \exp(-t^2).dt$ which is tabulated by Fried & Conte (1961).

$$\zeta_1 = \frac{\omega+\omega_H}{2^{\frac{1}{2}}.K_z v_T}, \quad \zeta_2 = \frac{\omega-\omega_H}{2^{\frac{1}{2}}.K_z v_T}, \quad \zeta_3 = \frac{\omega}{2^{\frac{1}{2}}K_z v_T}$$

$$\omega_p^2 = \frac{4\pi n_0 e^2}{m}, \quad \omega_H = \frac{eH}{mc}$$

Here the Maxwellian velocity distribution with the thermal velocity v_T and the variation $\sim e^{i(K_z z-\omega t)}$ are assumed. In using the local value of the di-electric tensor the restriction $KL << 1$ is imposed, K being the wave number and L the characteristic scale length of the plasma density variation. This condition is automatically satisfied for axially propagating quasi-static wave, i.e. for $K_r = 0$ In the quasi-static approximation $\vec{E}$ is derivable from a scalar potential, i.e.

$$\vec{E} = -\nabla\varphi$$

Hence the equation $\nabla.\vec{D} = 0$ becomes

$$\nabla\cdot(\bar{\bar{\epsilon}}\nabla\varphi) = 0 \quad \ldots \quad (3)$$

For the axi-symmetric mode, the equation (3) becomes in the cylindrical polar co-ordinates

$$\frac{d^2\varphi}{dr^2}+\frac{1}{r}\cdot\frac{d\varphi}{dr}+\frac{d}{dr}(\ln\epsilon_{rr})\cdot\frac{d\varphi}{dr}-\frac{\epsilon_{zz}}{\epsilon_{rr}}\cdot K_z^2\varphi = 0 \quad \ldots \quad (4)$$

Or,

$$u[1+\tau f(u)]\frac{d^2\varphi}{du^2}+\left[1+\tau f(u)+u\tau\frac{df(u)}{du}\right]\frac{d\varphi}{du}-u[1+\tau_0 f(u)]\varphi = 0$$

where $u = rK_z$ and $f(u) = 1-\frac{u^2}{\lambda_1^2 K_z^2} = 1-\frac{u^2}{\lambda^2}$.

This is a Sturm-Liouville equation which is here solved by the variational method, outlined by Margenau and Murphy (1966). The trial solution is taken as

$$\varphi = \varphi_0+\tfrac{1}{2}\varphi_1(K_z^2b^2-u^2) =\varphi_0+\tfrac{1}{2}\varphi_1(\delta^2-u)$$

so that

$$\varphi\,|_{r=b} = \varphi_0,$$

φ_0 and φ_0 are related by the condition

$$\frac{dk(\varphi)}{d\varphi_1} = 0 \qquad \ldots \quad (5)$$

where

$$k(\varphi) = \int_0^\delta [1+\tau f(u)]\varphi_1^2u^3du+\int_0^\delta [1+\tau_0 f(u)]\cdot\left[\varphi_0^2+\frac{1}{4}\varphi_1^2(\delta^2-u^2)^2+\varphi_0\varphi_1(\delta^2-u^2)\right]udu$$

Outside the plasma column the equation (4) reduces to

$$\frac{d^2\varphi}{du^2}+\frac{1}{u}\cdot\frac{d\varphi}{du}-\varphi = 0$$

the solution of which is

$$\varphi = AI_0(u)+BK_0(u)$$

where I_0 and K_0 are modified Bessel functions of order zero and, A and B are constants. At

$$u = ak_z = \Delta, \quad \frac{d\varphi}{du} = 0$$

Hence in the vacuum region

$$\varphi = \frac{A}{K_0(\Delta)}[K_0(\Delta)I_0(u)-I_0(\Delta)K_0(u)]$$

Again at $u = \delta$,

$$\varphi_{plasma} = \varphi_{vacuum}$$

$$\epsilon_{rr}(\delta)\cdot\frac{d\varphi_{plasma}}{du} = \frac{d\varphi_{vacuum}}{du}$$

These boundary conditions along with the equation (5) give the dispersion relation as

$$[K_0(\Delta)I_0(\delta)-I_0(\Delta)K_0(\delta)]\left(1+\tau-\tau\cdot\frac{\delta^2}{\lambda^2}\right)\cdot\delta\cdot\left[\frac{1}{4}(1+\tau_0)-\frac{1}{12}\tau_0\frac{\delta^2}{\lambda^2}\right]$$

$$= [K_0(\Delta)I_0'(\delta)-I_0(\Delta)K_0'(\delta)]\left[\frac{1}{2}(1+\tau)-\frac{1}{3}\tau\cdot\frac{\delta^2}{\lambda^2}+\frac{1}{12}(1+\tau_0)\delta^2-\frac{1}{48}\tau_0\delta^2\cdot\frac{\delta^2}{\lambda^2}\right] \qquad \ldots \quad (6)$$

Discussion

For a completely filled wave-guide $\delta = \Delta$, and the equation (6) becomes

$$\frac{1}{2}(1+\tau)-\frac{1}{3}\tau\epsilon^2+\frac{1}{12}(1+\tau_0)\,.\,\delta^2-\frac{1}{48}\,.\,\tau_0\delta^2\,.\,\epsilon^2=0 \qquad \dots (7)$$

where $\epsilon = \delta/\lambda$ determines the plasma density at the guide-wall. Taking zero density at the guide-wall ($\epsilon = 1$) the equation (7) becomes

$$\delta^2 = -6.\ \frac{1+\tau/3}{1+3\tau_0/4} \qquad \dots (8)$$

For a homogeneous plasma ($\epsilon = 0$) in a cylindrical waveguide the equation (7) becomes

$$\delta^2 = -6\ .\ \frac{1+\tau}{1+\tau_0} \qquad \dots (9)$$

Case 1. For cold plasma

$$\tau = \frac{\omega_p{}^2}{\omega_H{}^2-\omega^2} \quad \text{and} \quad \tau_0 = -\frac{\omega_p{}^2}{\omega^2}$$

Taking $\omega_H/\omega_p = 2$ the relations (8) and (9) are plotted in figure 1. It is found that the stop-band is wider in the case of inhomogeneous plasma.

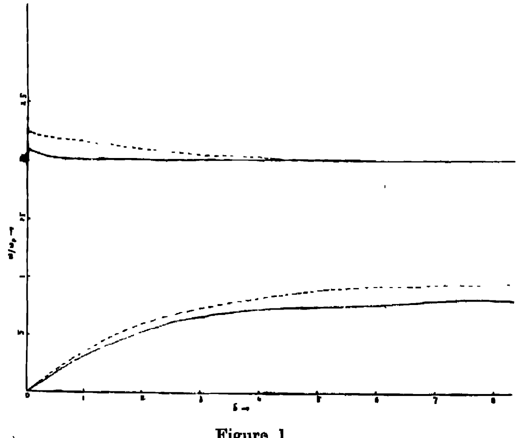

Figure 1

Case 2. For hot plasma, taking $\omega_H/\omega_p = 2$ and $v_T/a\omega_p = .072$ the relations (8) and (9) are plotted in figure 2. It is found that as in the case of cold plasma, the values of K_z for the inhomogeneous hot plasma are slightly higher than for the homogeneous hot plasma, the difference increasing towards the higher frequencies.

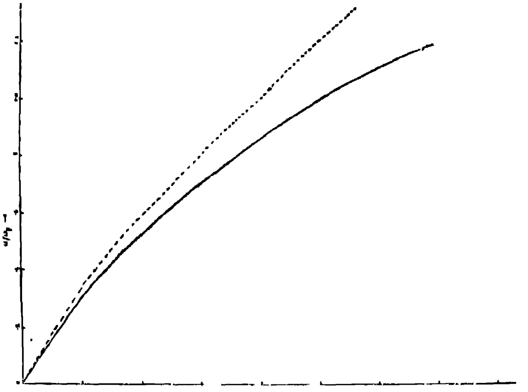

Figure 2

Case 3. Taking $\Delta = \infty$ the dispersion relation (6) becomes

$$\frac{1}{2} K_0(\delta)\,\delta\left(1+\frac{2}{3}\tau_0\right) = K_0'(\delta)\left[\left(1+\frac{1}{3}\tau\right)+\frac{1}{6}\left(1+\frac{3}{4}\tau_0\right).\delta^2\right] \text{ for } \epsilon = 1$$

and

$$\frac{1}{2}\,\delta.K_0(\delta).(1+\tau)(1+\tau_0) = K_0'(\delta).\left[(1+\tau)+\frac{1}{6}(1+\tau_0)\delta^2\right] \text{ for } \epsilon = 0$$

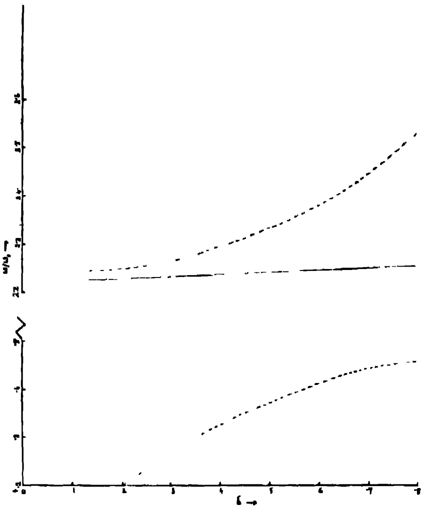

Figure 3

For $\delta < 1$ the Bessel functions are replaced by their series expansions arrested to the second power of δ and then these relations are plotted in figure 3. It is seen that in the range $0 < \delta < 1$ any value of δ corresponds to two values of ω_p for the homogeneous cold plasma cylinder, the upper one being slightly higher than the only value of ω/ω_p for the inhomogeneous cold plasma cylinder.

In all the figures the solid line represents the inhomogeneous plasma while the broken line represents the homogeneous plasma.

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